IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 09, Issue 2 (February. 2019), ||S (II) || PP 38-42

A New Class of Analytic and Multivalent Function Associated With a Fractional Calculus Operator

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Abstract: In this paper, we introduce a new class $W_k(a, b, \alpha, \beta)$ consists of multivalent function which is analytic in the open unit disk with negative coefficient defined with the help of Hohlov operator. Characterization property, distortion theorems and some other interesting results of this class are investigated.

Keywords: Distortion Theorem, Generalized hypergeometric Function, Hohlov Operator, Hadmard Product.

Date of Submission: 28-01-2019 Date of acceptance:11-02-2019

I. INTRODUCTION

Let A denote the class of functions defined as (1), (2)

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n \tag{1}$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n\, z^n \tag{2}$$

which are analytic and univalent in open unit disk U = {z:$|z| < 1$} is in A.
Then convolution or Hadmard product of *f*(z) and *g(z)* is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n\, b_n z^n \qquad z \in U. \tag{3}$$

Let $W_k(a, b, \alpha, \beta)$ denote the subclass of A consisting of functions of the form

$$f(z) = z^k - \sum_{n=2}^{\infty} a_{n+k-1}\, z^{n+k-1}, (a_{n+k-1} > 0: k \in \mathbb{N}) \tag{4}$$

which are analytic and k-valent in the open unit disk U = {z: zϵℂ,$|z| < 1$}.

Definition1. Generalized hypergeometric function ${}_pF_q$ is defined in [5] as follows for $a_1, a_2 \ldots\ldots a_p, b_1, b_2 \ldots\ldots b_q \in \mathbb{N}$

$$\Rightarrow {}_pF_q[a_1, a_2 \ldots\ldots a_p; b_1, b_2 \ldots\ldots b_q; z] = 1 + \sum_{n=2}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_{n-1}\, z^{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!} \tag{5}$$

$$(\mathrm{p} \leq \mathrm{q} + 1), \qquad |z| < 1$$

where $(a)_n$ is pochhammer symbol defined, in terms of the Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{Bmatrix} 1, if\ (n = 0) \\ a(a+1) \ldots . (a+n-1)\,, if\ (n \in \mathbb{N}) \end{Bmatrix}.$$

Definition 2: Let $f(z)\epsilon(a, b, \alpha, \beta)$ be of the form (4) then the Hohlov operator $(\big(I(a, b)f(z)\big), [(I(a, b)f(z)): W_k \rightarrow W_k)]$ [4] is defined by means of a Hadmard product below:

$$\big(I(a, b)f(z)\big) = z^k\, {}_pF_q[a_1, a_2 \ldots\ldots a_p; b_1, b_2 \ldots\ldots b_q; z] * f(z) = z^k - \sum_{n=2}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_{n-1}\, z^{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!} a_{n+k-1}\, z^{n+k-1}$$

$$(a_1, a_2 \ldots\ldots a_p, b_1, b_2 \ldots\ldots b_q \in \mathbb{N}\ \&\ k \in \mathbb{N}, z \in \mathrm{U}). \tag{6}$$

Definition 3: A function *f(z)* is in the class $W_k(a,b,\alpha,\beta)$ if it satisfies the condition

$$\left|\frac{z^{2-k}\big(I(a,b)f(z)\big)''}{z^{2-k}\big(I(a,b)f(z)\big)''-2(1-\alpha)z^{1-k}\big(I(a,b)f(z)\big)'}\right|<\beta.$$

where $0\le\alpha<1, 0<\beta\le 1, k\in\mathbb{N}, z\epsilon U.$ (7)

II. CHARACTERIZATION PROPERTY

We now investigate the characterization property for the function *f*(z) belongs to the class $W_k(a,b,\alpha,\beta)$ thereby, obtaining the coefficient bound.

Theorem 1: Let the function *f* be defined by (4) then $f(z)\epsilon W_k(a,b,\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]a_{n+k-1} \le k[(k-1)(1-\beta)+2\beta(1-\alpha)]$$

where $0\le\alpha<1, 0<\beta\le 1, a_{n+k-1}>0, k\in\mathbb{N}$ (8)

the result is sharp for the below function.

$$f(z)=z^k-\frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}{(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p}(a_i)_{n-1}}z^{n+k-1}$$

Proof : Suppose that the inequality (8) holds true and let $|z|<1$. Then we obtain

$$\left|z^{2-k}\big(I(a,b)f(z)\big)''\right|-\beta\left|z^{2-k}\big(I(a,b)f(z)\big)''-2(1-\alpha)z^{1-k}\big(I(a,b)f(z)\big)'\right|$$

$$=\left|k(k-1)-\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)(n+k-2)a_{n+k-1}\right|$$

$$-\beta\left|k[(k-1)-2(1-\alpha)]-\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)\,a_{n+k-1}[(n+k-2)-2(1-\alpha)]\right|$$

$$\le\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]a_{n+k-1}-k[(k-1)(1-\beta)+2\beta(1-\alpha)]$$

≤ 0

by our hypothesis. Hence by the maximum modulus principle $f(z)\epsilon W_k(a,b,\alpha,\beta)$.

To prove the converse, assume that *f*(z) is defined by (4) and is in the class$W_k(a,b,\alpha,\beta)$ so by the condition (7).

$$\left|\frac{z^{2-k}\big(I(a,b)f(z)\big)''}{z^{2-k}\big(I(a,b)f(z)\big)''-2(1-\alpha)z^{1-k}\big(I(a,b)f(z)\big)'}\right|<\beta$$

$$\Rightarrow\left|k(k-1)-\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)(n+k-2)a_{n+k-1}z^{n-1}\right|.$$

$$\left|k(k-1)-\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)(n+k-2)a_{n+k-1}z^{n-1}-2(1-\alpha)\left\{k-\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)\,a_{n+k-1}z^{n-1}\right\}\right|^{-1}<\beta$$

$$\Rightarrow\left|\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)(n+k-2)a_{n+k-1}z^{n-1}-k(k-1)\right|.$$

$$\left|\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)a_{n+k-1}z^{n-1}[(n+k-2)-2(1-\alpha)]-k(k-1)+2k(1-\alpha)\right|^{-1}<\beta$$

(9)

since $|Re(z)|\le|z|$ for any z, we find from (9) that

$Re\left\{\left[\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)(n+k-2)a_{n+k-1}z^{n-1}-k(k-1)\right].\ \left[\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+\right.\right.$

k−1an+k−1zn−1n+k−2−21−α−kk−1+2k(1−α)−1<β

(10)

Choose values of *z* on the real axis so that $(I(a,b)f(z))$is real. Taking $z\to 1$ through real values, we have

$$\Rightarrow\left[\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)\,(n+k-2)a_{n+k-1}\,z^{n-1}-k(k-1)\right]$$
$$\leq\beta\left[\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}(n+k-1)a_{n+k-1}z^{n-1}[(n+k-2)-2(1-\alpha)]-k(k-1)\right.$$
$$\left.+2k(1-\alpha)\right]$$

$\Rightarrow\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}\,(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]a_{n+k-1}\leq k[(k-1)(1-\beta)+$
$2\beta(1-\alpha)]$

desired assertion (8) is proved .

Finally, we note that the assertion (8) of theorem 1 is sharp for the function

$$f(z)=z^k-\frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}{(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p}(a_i)_{n-1}}z^{n+k-1}$$

(11)

Corollary 1. Let the function *f*(z) defined by (4) belong to the class $W_k(a,b,\alpha,\beta)$ then

$$a_{n+k-1}\leq\frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}{(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p}(a_i)_{n-1}},n\epsilon\mathbb{N}$$

(12)

if we put k=1, i=1, j=1, b=1, $\alpha=\gamma$, then its reduced to [1]

$$a_n\leq\frac{2\beta(1-\gamma)}{(n)[(n-1)(1-\beta)+2\beta(1-\gamma)].a}$$

if n=2 then [1]

$$a_2\leq\frac{\beta(1-\gamma)}{[1+\beta(1-2\gamma)].a}.$$

Theorem 2: Let the function f(z) be defined by (4) and g(z) defined by

$$g(z)=z^k-\sum_{n=2}^{\infty}b_{n+k-1}\,z^{n+k-1}$$

$(b_{n+k-1}>0:k\epsilon\mathbb{N})$ (13)

be in the class $W_k(a,b,\alpha,\beta)$, then the function *h(z)* is defined by

$$h(z)=\ (1-\theta)f(z)+\theta g(z)=z^k-\sum_{n=2}^{\infty}c_{n+k-1}\,z^{n+k-1}$$
$$(c_{n+k-1}=(1-\theta)a_{n+k-1}+\theta b_{n+k-1}\,;0\leq\theta<1,k\in\mathbb{N})\qquad(14)$$

is also in the class $W_k(a,b,\alpha,\beta)$.

Proof: By the hypothesis of theorem 2 we find from (8) that

$$\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}\,(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]c_{n+k-1}$$
$$=\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}\,(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)](1-\theta)a_{n+k-1}$$
$$+\sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}\,(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]\theta b_{n+k-1}$$
$$\leq(1-\theta)\,k[(k-1)(1-\beta)+2\beta(1-\alpha)]+\theta\,k[(k-1)(1-\beta)+2\beta(1-\alpha)]$$
$$\leq k[(k-1)(1-\beta)+2\beta(1-\alpha)]$$

Hence the function *h*(z) satisfies the condition (7) & $h(z)\in W_k(a,b,\alpha,\beta)$.

III. DISTORTION THEOREM

Following [8] we can easily prove the results given below:

Theorem 3: Let $0 \leq \alpha < 1, 0 < \beta \leq 1, k \in \mathbb{N}$ .If the function *f*(z) defined by (4) being the class $W_k(a, b, \alpha, \beta)$ then

$$|f(z)| \geq |z|^k - \frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q} b_j}{(k+1)[(k)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p} a_i}|z|^{k+1} \tag{15}$$

$$|f(z)| \leq |z|^k + \frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q} b_j}{(k+1)[(k)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p} a_i}|z|^{k+1} \tag{16}$$

Proof :- Since $f(z) \epsilon W_k(a, b, \alpha, \beta)$ By theorem 1 we have

$$\frac{(k+1)[(k)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p} a_i}{\prod_{j=1}^{q} b_j}\sum_{n=2}^{\infty} a_{n+k-1}$$

$$\leq \sum_{n=2}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{n-1}}{\prod_{j=1}^{q}(b_j)_{n-1}(n-1)!}\,(n+k-1)[(n+k-2)(1-\beta)+2\beta(1-\alpha)]a_{n+k-1}$$

$$\leq k[(k-1)(1-\beta)+2\beta(1-\alpha)]$$

$$\Rightarrow \sum_{n=2}^{\infty} a_{n+k-1} \leq \frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q} b_j}{(k+1)[(k)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p} a_i}$$

Consequently, we obtain $|f(z)| \geq |z|^k - |z|^{k+1}\sum_{n=2}^{\infty} a_{n+k-1}$

$$|f(z)| \geq |z|^k - \frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q} b_j}{(k+1)[(k)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p} a_i}|z|^{k+1}$$

and

$$|f(z)| \leq |z|^k + |z|^{k+1}\sum_{n=2}^{\infty} a_{n+k-1}$$

$$|f(z)| \leq |z|^k + \frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)]\prod_{j=1}^{q} b_j}{(k+1)[(k)(1-\beta)+2\beta(1-\alpha)]\prod_{i=1}^{p} a_i}|z|^{k+1}$$

Which prove the assertions (15), (16) of theorem 3.

Corollary 2- Under the hypothesis of theorem 3, *f*(z) is included in an open unit disk with its center at the origin and radius r given by:

$$r = 1 + \frac{k[(k-1)(1-\beta)+2\beta(1-\alpha)].\prod_{j=1}^{q} b_j}{(k+1)[k(1-\beta)+2\beta(1-\alpha)].\prod_{i=1}^{p} a_i}$$

IV. FURTHER PROPERTIES OF $W_k(a, b, \alpha, \beta)$

Now we study some interesting properties of the class $W_k(a, b, \alpha, \beta)$. The proof of each of the following results in this section runs parallel to that of the corresponding assertion made by Srivastava and Aouf [11]. We skip details involved.

Theorem 4: Let the conditions given by (4) be satisfied and $0 \leq \alpha' < 1$, $0 < \beta' \leq 1$, then
$W_k(a, b, \alpha, \beta) = W_k(a, b, \alpha', \beta')$,
If and only if $\frac{\beta(1-\alpha)}{(1-\beta)+2\beta(1-\alpha)} = \frac{\beta'(1-\alpha')}{(1-\beta')+2\beta'(1-\alpha')}$ (17)

Theorem 5: Let $0 \leq \alpha_1 \leq \alpha_2 < 1$ *and* $0 < \beta \leq 1$, then
$W_k(a, b, \alpha_1, \beta) \supseteq W_k(a, b, \alpha_2, \beta)$ (18)

Theorem 6: Let $0 < \beta_1 \leq \beta_2 \leq 1$ *and* $0 \leq \alpha < 1$, then
$W_k(a, b, \alpha, \beta_1) \subseteq W_k(a, b, \alpha, \beta_2)$ (19)

REFERENCES

[1]. Atshan W. G. (2010), Application of fractional calculus operators for a new class of univalent functions with negative coefficients defined by Hohlov operator, *Math. Slovaca* 60 No. 1, 75–82.

[2]. Atshan W. G., Majeed A.H. and Jassim K.A. (2014), On a new class of univalent functions with application of fractional calculus operators Defined by Hohlov operator, Journal of Asian scientific research, 4(2):99-111.

[3]. Gottschalk J E and Maslen E N(1988), Reduction formulae for generalized hypergeometric functions of one variable, *J. Phys. A: Math. Gen*. 21 1983-1998. Printed In The UK.

[4]. Hohlov Y.E. (1989), Convolution operators preserving univalent functions, PLISKA 10, 87-92.
[5]. Mishra A. K.. and Panigrahi Trailokya(2011), Class-Mapping properties of the Hohlov operator, *Bull. Korean Math. Soc.* 48 No. 1, Pp. 51-65 ,DOI 10.4134/BKMS.2011.48.1.051.
[6]. Omar R., Halim S. A. and Janteng A., (2017), Subclasses of Bi-univalent functions associated with Hohlov operator, World Academy of scirnce, Engineering and Technology, *International journal of Mathematical and Computational Sciences*, Vol:11, No:10, 465-468.
[7]. Owa, S. (1978), On the distortion theorems, *Kyungpook Math. J.* 18, 53–59.
[8]. Raina R.K. and Choi J. H.(2002), On a subclass of analytic and multivalent functions associated with certain fractional calculus operator , *Indian J. Pure Appl. Math.* 33(1), 55-62.
[9]. Raina R. K. and Srivastava H. M. (1999), Some subclasses of analytic functions associated with fractional calculus operators, *Computers and Mathematics with Applications* 37, 73-84.
[10]. Sivasubramanian S., Rosy T., Muthunagai K.(2011), Certain sufficient conditions for a subclass of analytic functions involving Hohlov operator, *Computers and Mathematics with applications* 62, 4479-4485.
[11]. Srivastava H. M. and Aouf M. K. (1992), A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients, *Journal of Mathematical Analysis and Applications* 171,1-13

IOSR Journal of Engineering (IOSRJEN) is UGC approved Journal with Sl. No. 3240, Journal no. 48995.

I.B.Bapna. "A New Class of Analytic and Multivalent Function Associated With a Fractional Calculus Operator." IOSR Journal of Engineering (IOSRJEN), vol. 09, no. 02, 2019, pp. 38-42.